Everything Bob Knows About Limits

The 21st century world is built on the foundational blocks that Calculus mined and prepared. Its importance to our civilization can not be over-estimated.

So what is Calculus? A common definition (or sound-byte really) is that Calculus is the study of change – and that is correct if understood properly. In the Calculus of Derivatives that we are studying, the underlying goal or theme is to discover how the values of a function behave when the value of x is close to a specific value.

The study of limits actually came after Calculus was pretty well established – although the idea of a limiting process was crucial to the development of Calculus. The study of limits is a by-product of the revolution in the 19th century that turned modern mathematics into a rigorous system of thinking in the style of Euclid.

We study limits today for two reasons:

* They are how we measure the 'predictability' of a function – in an idea we now define as 'continuity'. Remember – polynomials incorporate ALL the nice ideas that we want in the functions we use to model the world. So in some ways, Calculus is the science of identifying the polynomial properties that we like the most, and discovering what other types of functions have the same property. The study of limits is crucial to this discovery.

* They underlie the major discoveries of Calculus: Derivatives (and the limit definition of the derivative in all its moods is extremely important), and Integrals (where limits show themselves in the study of Riemann Sums).

<u>Evaluating Limits:</u>

► If a function is continuous at a point x=a, then the value of the limit (as x->a) = f(a). (This follows from the definition of continuous … which you should know!).

The functions that are continuous at every point in their domain are:

any polynomial any exponential function any logarithmic function
any trigonometric function any rational function

and we learned later that any differentiable function is continuous.

This is why we try substitution first.

Now for the difficulties. Remember that limits ask what the value of the function ***SHOULD BE*** as we approach a specific *x-value*. It does not matter what the actual value of the function is at that specific *x-value,* or even if that *x-value* is in the domain of *f.*

We need to deal with two specific complications:

1) The function is not continuous at a specific *x-value*
2) We want the limit as x approaches $\pm\infty$ (what we called the end behavior in PreCalc).

► ***Rational Functions***

1) If the function is not continuous at x=a, then substituting the value 'a' into the polynomial in the denominator makes it 0. There are then two possibilities:

i) The numerator is ALSO 0 at x=a. In this case *(x-a)* is a factor of top and bottom and can be cancelled. Do so, and try again with the simplified rational function.

ii) The numerator is NOT 0 at x=a. In this case, you need to consider the two one-sided limits: x->a+ and x->a-. The limit will either not exist (period), or it won't exist because it tends to $+\infty$ or $-\infty$.

► ***Rational Functions (continued)***

2) Rational functions and limits to infinity

Let m be the highest degree in either polynomial (the one in the denominator or the one in the numerator). Multiply every term in numerator and denominator by $\frac{1}{x^m}$. This will turn only one term in the top or bottom (or both) into a number – all the other terms will have an x in the denominator (and so their limits will be 0). If the denominator is approaching 0, consider the signs of the terms. Your limit will approach $+\infty$ or $-\infty$.

► ***Limits as x approaches 0 in a denominator***

A general note of warning. Everything on the number line behaves kind of rationally (HA HA! Get it? I crack myself up …). With one exception – consider $\lim_{x\to 0}\frac{1}{x}$. Unfortunately, there are two ways to 'sneak up' on the 0 in the denominator: from the left (all negative numbers) and from the right (all positive numbers). If we approach 0 from the left; the limit approaches negative infinity (the graph of 1/x will illustrate nicely what is happening). If we approach 0 from the right; the limit approaches positive infinity. And so the function, Janus-like … facing both higher positive numbers and lower negative numbers, has no two-sided limit. **You need to consider the two one-sided limits of any function that has a denominator approaching 0.**

► ***Radicals in functions***

One technique that you should always consider with radicals is multiplying by a conjugate. We've seen problems where this is the correct strategy even when the radical is not in a fraction. (Consider the following limit: $\lim_{x\to\infty}\sqrt{x+1}-\sqrt{x}$)

Another complication involving radicals is when they are in a fraction, but conjugates don't help (these primarily occur when the limit is to infinity). Consider as an example: $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+4}}$. The top and bottom both approach infinity. The strategy here is to multiply the top and bottom by $\frac{1}{x}$. Easy on the top – but what do you do with the radical on the bottom? Well – you have to change $\frac{1}{x}$ to $\frac{1}{\sqrt{x^2}}$. Then you can multiply through the radical: $\sqrt{x^2+4}\cdot\frac{1}{\sqrt{x^2}}=\sqrt{\frac{x^2}{x^2}+\frac{4}{x^2}}=\sqrt{1+\frac{4}{x^2}}$ and this limit is 1. (Remember – this is in the denominator above).

► ***Trigonometric functions***

Remember that limits might not exist because of a never-ending oscillation (as in $\lim_{x\to\infty}\sin(x)$).

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0$$

Consider this – and then let me know what other questions you still have. And maybe we'll have a quiz then ☺